THE 28th ANNUAL (2006) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend or write stray marks on either side of the answer sheet. Each correct answer is worth 4 points. **Two points are deducted for each incorrect answer.** *Zero points are given if no box, or more than one box, is marked. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to the proctor. You may keep your copy of the questions.*

NO CALCULATORS

75 MINUTES

1. A box of coins contains a total of \$26.00 in nickels, dimes, and quarters. If there is the same number of nickels as dimes, but twice as many quarters as nickels, how many dimes are in the box?
   a. 26 b. 30 c. 34 d. 40 e. 50

2. Mickey cannot afford to buy Minnie a diamond, so he draws her a picture of one instead. He graphs the set D defined by $|3x| + |2y| \leq 12$. What is the area of D?
   a. 24 b. 36 c. 48 d. 96 e. 144

3. Let a be a real number. The system of equations $3x + 2y = 8$ and $ax - 8y = 9$ has no solutions (x, y). What is the value of a?
   a. 0 b. 1 c. 3 d. -8 e. -12

4. Which of the following is the largest?
   a. $2^{1/2}$ b. $5^{1/3}$ c. $8^{1/4}$ d. $11^{1/5}$ e. $14^{1/6}$

5. A child has 100 plastic figurines, some of which are 4-legged dinosaurs and some of which are 2-legged dinosaurs. If there is a total of 260 legs, how many 4-legged dinosaurs are in the collection?
   a. 30 b. 50 c. 65 d. 70 e. 85

6. The parabolas $y = x^2$ and $y = 2x^2 - 3x + 2$ intersect in the points P and Q. What is the slope of the straight line through P and Q?
   a. -1 b. 0 c. 2 d. 3 e. -3

7. If x satisfies $4\log_2(x) = \log_2(81)$, then $x =$
   a. 3 b. 9 c. 16 d. 81/4 e. 81/16

8. In quadrilateral $ABCD$, angles ABC and DAC are right angles. If side AB has length 4, side BC has length 3, and side AD has length 12, how long is side CD?
   a. 11 b. 12 c. 13 d. 15 e. 16

9. Manet can paint a house in 3 hours. Monet can paint the house in 2 hours. If they work together, how many hours will it take them to paint the house?
   a. 1 b. 1.2 c. 1.25 d. 1.4 e. 1.5

10. Winnie the Pooh and his friends are standing 200 feet from a tree. Pooh walks over to the tree and climbs it, trying to get some honey. He gets stuck at a height H above the ground. His friends notice that they have to look up at an angle of 15° above horizontal to see Pooh. Since Christopher Robin recently explained trigonometry to them, they try to calculate H. Rabbit computes $200 \cos 15^\circ$. Kanga computes $200 \sin 15^\circ$. Eeyore computes $200 \tan 15^\circ$. Piglet computes $100 \tan 30^\circ$. Roo computes $600 \sin 5^\circ$. Who is correct?
a. Rabbit b. Kanga c. Eeyore d. Piglet e. Roo

11. Let $A = 2^0 + 2^1 + \cdots + 2^{1000}$. If you divide A by 7, what is the remainder?
a. 0 b. 1 c. 2 d. 3 e. 4

12. Euclid asks his friends to guess the value of a positive integer n that he has chosen. Archimedes guesses that n is a multiple of 10. Euler guesses that n is a multiple of 12. Fermat guesses that n is a multiple of 15. Gauss guesses that n is a multiple of 18. Hilbert guesses that n is a multiple of 30. Exactly two of the guesses are correct. Which persons guessed correctly?
a. Fermat and Gauss b. Archimedes and Hilbert c. Gauss and Hilbert d. Euler and Gauss
e. Archimedes and Fermat

13. In a school of 150 students, 60 have had chicken pox, 50 have had measles, and 40 have had mumps. Also, 30 have had chicken pox and measles, 15 have had chicken pox and mumps, while 10 have had measles and mumps. If 5 students have had all three diseases, how many students have had none of the three?
a. 0 b. 10 c. 15 d. 35 e. 50

14. Two trains face each other on the same track, 100 miles apart. Train A starts moving toward B at 10 miles per hour, while train B starts moving toward A at 30 mph. A fly starts on Train A's headlight and flies toward B at a rate of 70 mph. After it hits B, it turns around, flies back to A, again at 70 mph. When it reaches A, what is the total number of miles that the fly has traveled?
a. 70 b. 87.5 c. 100 d. 122.5 e. 140

15. Let $f(x, y)$ be defined by (i) $f(x, 0) = x$, and (ii) $f(x, y + 1) = f(f(x, y), y)$. Which of the following is the largest?
a. $f(10, 15)$ b. $f(11, 14)$ c. $f(12, 13)$ d. $f(13, 12)$ e. $f(14, 11)$

16. A motor boat takes 5 hours to go from port A to port B on the Mississippi river and 7 hours to go back. How many hours will it take for a raft to go from A to B? The motor boat travels at a constant speed relative to the water. The raft has no motor and therefore travels at the same speed as the water.
a. 25 b. 6 c. 12 d. 35 e. 13.5

17. Exactly one of the digits of an integer n is 0. Deleting this zero yields an integer r with $9r = n$. What is the number of digits in the largest such n?
a. 2 b. 3 c. 4 d. 5 e. 6

18. What is the number of solutions of the equation $\sin x = x/100$? (the real number x is in radians)
a. 100 b. 71 c. 49 d. 63 e. infinite

19. Start with the integers from 1 to 10^{2006}. Replace each of them by the sum of its digits to get a string of 10^{2006} numbers. Keep doing this until you get 10^{2006} single digit numbers. Let m be the number of 1's and n be the number of 2's. Then $m - n$ equals which of the following?
a. 0 b. 1 c. 2 d. 3 e. None of the preceding.

20. Consider the variant of tic-tac-toe (still played on a 3x3 board) with the stipulation that if you have a chance of making 3 in a row on your move, you MUST do so, but then the game ends and you LOSE. Which of the following statements is true?
I. If the first player's first move is in the center, the second player can guarantee a win.
II. If the first player's first move is in a corner and the second player's first move is in the center, then the first player can guarantee a win.
III. If the first player's first move is in a corner and the second player's first move is in the diagonally opposite corner, then the second player can guarantee a win.
a. I and III b. II and III c. I and II d. I only e. II only

21. Circles C_1 and C_2 lie in the plane. C_1 has center at A and radius 5, and C_2 has center at B and radius 7. The distance from A to B is 10. The two circles intersect in points P and Q. Let L be a line through P that intersects C_1 at P and E and intersects C_2 at P and F. Let ℓ be the longest possible length for EF. Then
a. $\ell \leq 19$ b. $19 < \ell \leq 20$ c. $20 < \ell \leq 21$ d. $21 < \ell \leq 22$ e. $22 < \ell$

22. Let p, q, r, s, t be the numbers $1, 2, 3, 4, 5$, but not necessarily in this order (for example, we could have $p = 4, q = 2, r = 5, s = 3, t = 1$). Let

$$x = \cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{s + \frac{1}{t}}}}}.$$

If you choose p, q, r, s, t so that x is as large as possible, then $t =$
a. 1 b. 2 c. 3 d. 4 e. 5

23. The sides a, b, c of a triangle satisfy $0 \leq a \leq 1 \leq b \leq 2 \leq c \leq 3$. What is the largest possible area of such a triangle?
a. 1 b. 1/2 c. 2/3 d. 2 e. None of the preceding.

24. Let

$$x = \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{999}{1000!}.$$

Which of the following inequalities is true?
a. $x < 0.999$ b. $0.999 \leq x < 1 - 10^{-12345}$ c. $1 - 10^{-12345} \leq x < 1.0$ d. $1.0 \leq x < 1 + 10^{-12345}$ e. $1 + 10^{-12345} \leq x$

25. The value of

$$\frac{1}{2 \sin 10^\circ} - 2 \sin 70^\circ$$

is which of the following:
a. 1 b. $\sin 80^\circ$ c. $\sqrt{2}$ d. 0 e. $\sin 50^\circ$